MATH 263 - FALL 2010

(By S.K. and K.J)

1) a) $$y' + \frac{y}{(1+x)x} = x^2(1+x).$$

IF: $$\mu(x) = \exp\left(\int \frac{1}{x(1+x)}\,dx\right) = e^{\left(\int \frac{1}{x} - \frac{1}{1+x}\,dx\right)}$$

$$= \exp\left(\ln|x| - \ln|1+x|\right) = e^{\ln\left(\frac{x}{1+x}\right)} = \frac{x}{1+x}$$

(we can remove the absolute values under certain conditions: $x>0$).

so $\left(\frac{x}{1+x}\,y\right)' = x^3$ so $\frac{x}{1+x}\,y = \int x^3\,dx$.

and $$y = \frac{x^3(1+x)}{4}.$$

b) $y' = y(3 - xy)$

$y' - 3y + xy^2 = 0 \Rightarrow$ Bernoulli equation with $n=2$.

set $u = y^{1-2} = y^{-1} = 1/y$.

$u' = -y^{-2}y'$ so $y' = -\frac{u'}{y^{-2}} = -\frac{u'}{u^2}$.

substituting in the equation: $-\frac{u'}{u^2} = \frac{1}{u}\left(3 - \frac{x}{u}\right)$.

so $-\frac{u'}{u} = 3 - \frac{x}{u}$ and $-u' = 3u - x$.

so $u' + 3u = x$

$\mu(x) = e^{\int 3\,dx} = e^{3x}$

so $\left[e^{3x}u\right] = \int x e^{3x}\,dx \overset{\text{by parts}}{=} \left[\frac{xe^{3x}}{3}\right] - \int \frac{e^{3x}}{3}\,dx = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9}$.

so $u = \frac{x}{3} - \frac{1}{9} = \frac{3x-1}{9}$ and $y = \frac{9}{3x-1}$

2) a) $xy' + y = \cos(x)$

$y' + \frac{1}{x} y = \frac{\cos(x)}{x}$

so $\mu(x) = \exp \int \frac{1}{x} dx = e^{\ln|x|} = x$ (for $x > 0$).

so $(y \times x)' = \cos(x)$ and $y \times x = \int \cos x \, dx = \sin x$.

so $y = \frac{\sin x}{x}$.

b) Let y_1 and y_2 satisfy the same linear ODE with $y_1 \neq y_2$.

therefore $\begin{cases} a_1(x) y_1' + a_2(x) y_1 = f(x) & (1) \\ a_1(x) y_2' + a_2(x) y_2 = f(x) & (2). \end{cases}$

Substracting (1) from (2) (or (2) from (1)),

we get the equation: $a_1(x)[y_1' - y_2'] + a_2(x)[y_1 - y_2] = 0$.

Defining $y = y_1 - y_2$, we have : $a_1(x) y' + a_2(x) y = 0$.

(by linearity of derivation).

Since $y_1 \neq y_2$, y is a non-zero solution of the homogeneous ODE.

The general solution to the original ODE would satisfy $a_1(x) Y' + a_2(x) Y = f(x)$.

take $Y(x) = \frac{y_1(x) + y_2(x)}{2}$

(It works: $a_1(x)\left(\frac{y_1'(x)}{2} + \frac{y_2'(x)}{2}\right) + a_2(x)\left(\frac{y_1(x)}{2} + \frac{y_2(x)}{2}\right) = \frac{f(x)}{2} + \frac{f(x)}{2} = f(x)$.

and it's unique in terms of y_1 and y_2).

3) a) $L = D^2 + cD + kI$, $w(t) = e^{-2t}\sin(t)$ for $t > 0$. $\begin{cases} w(0) = 0 \\ w'(0) = 0 \end{cases}$

$\mathcal{L}\{L(w(t))\} = s^2 W(s) + cs\,W(s) + k\,W(s)$ since $w(0) = w'(0) = 0$.

with $W(s) = \mathcal{L}\{w(t)\} = \mathcal{L}\{e^{-2t}\sin t\} = \frac{1}{(s+2)^2+1}$ so :

$\therefore \mathcal{L}\{L(w(t))\} = \frac{s^2+cs+k}{(s+2)^2+1}$

We want $L(w(t)) = \delta(t)$, plus $\mathcal{L}\{\delta(t)\} = 1$.

so $\frac{s^2+cs+k}{(s+2)^2+1} = 1$ so $s^2+cs+k = s^2+4s+5$.

so $c = 4$ and $k = 5$.

b) $L = D^2 + 4D + 5I$

$\mathcal{L}\{L[y]\} = Y(s)(s^2+4s+5)$ since $y'(0) = y(0) = 0$.

$\mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}$.

so $Y(s)(s^2+4s+5) = \frac{1}{s+2}$

so $Y(s) = \frac{1}{s+2} \times \frac{1}{s^2+4s+5} = \frac{1}{s+2} \times \frac{1}{(s+2)^2+1}$

so $y(t) = \mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s+2}}_{F(s)} \times \underbrace{\frac{1}{(s+2)^2+1}}_{G(s)}\right\}$ $\quad \mathcal{L}^{-1}\{F(s)\} = e^{-2t} = f(t)$

$\mathcal{L}^{-1}[G(s)] = e^{-2t}\sin t = g(t)$

so $y(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$

$y(t) = \int_0^t e^{-2(t-\tau)}\,e^{-2\tau}\sin\tau\,d\tau = \int_0^t e^{-2t}\sin\tau\,d\tau = e^{-2t}\left[-\cos t\right]_0^t$

$y(t) = e^{-2t}(1-\cos t)$

4) $y'' + 4y = \sin(t) - u_{2\pi}\sin(t-2\pi) \quad y(0) = y'(0) = 0$

a) $\mathcal{L}\{y'' + 4y\} = s^2 Y(s) + 4Y(s) = (s^2+4)Y(s)$.

$$\mathcal{L}\{\sin(t) - u_{2\pi}\sin(t-2\pi)\} = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}$$

so $$Y(s) = \frac{1}{(s^2+4)(s^2+1)} - \frac{e^{-2\pi s}}{(s^2+4)(s^2+1)}$$

$$\frac{1}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

with $s=0$: $\frac{B}{4} + D = \frac{1}{4}$. (1)

$s=1$: $\frac{A+B}{5} + \frac{C+D}{2} = \frac{1}{10}$ (2)

$s=-1$: $\frac{-A+B}{5} + \frac{-C+D}{2} = \frac{1}{10}$ (3)

so $\begin{cases} \frac{B}{4} + D = \frac{1}{4} \\ \frac{2B}{5} + D = \frac{1}{5} \end{cases}$ ((2)+(3)).

so using Cramer's rule: $B = \frac{\begin{vmatrix} 1/4 & 1 \\ 1/5 & 1 \end{vmatrix}}{\begin{vmatrix} 1/4 & 1 \\ 2/5 & 1 \end{vmatrix}} = -\frac{1}{3}$, $D = \frac{\begin{vmatrix} 1/4 & 1/4 \\ 2/5 & 1/5 \end{vmatrix}}{\begin{vmatrix} 1/4 & 1 \\ 2/5 & 1 \end{vmatrix}} = \frac{1}{3}$.

so substituting back in (2) and (3): $\frac{A}{5} + \frac{C}{2} = 0$

Plus with $s=2$: $\frac{2A+B}{8} + \frac{2C+D}{5} = \frac{1}{40}$ so $\frac{A}{4} + \frac{2C}{5} = 0$. (4)

so using Cramer's rule: $A = \frac{\begin{vmatrix} 0 & 1/2 \\ 0 & 2/5 \end{vmatrix}}{\begin{vmatrix} 1/5 & 1/2 \\ 1/4 & 2/5 \end{vmatrix}} = 0$ and $C = \frac{\begin{vmatrix} 1/5 & 0 \\ 1/4 & 0 \end{vmatrix}}{\begin{vmatrix} 1/5 & 1/2 \\ 1/4 & 2/5 \end{vmatrix}} = 0$

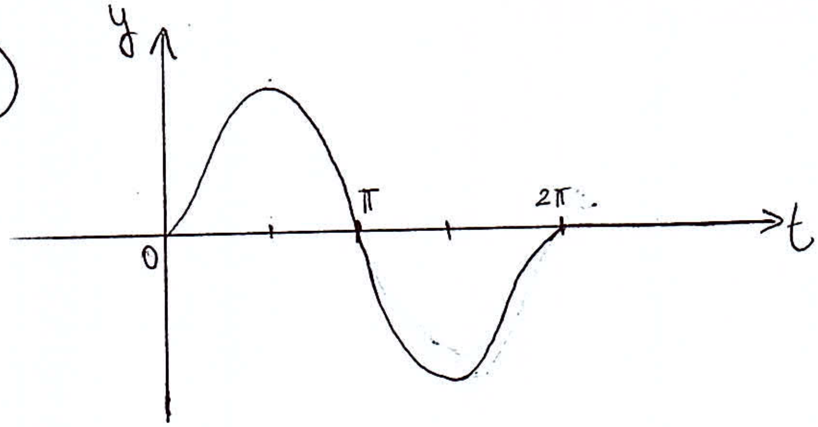
so $$\frac{1}{(s^2+4)(s^2+1)} = \frac{-1/3}{s^2+4} + \frac{1/3}{s^2+1}$$

so $Y(s) = \dfrac{-1/3}{s^2+4} + \dfrac{1/3}{s^2+1} - e^{-2\pi s}\left[\dfrac{-1/3}{s^2+4} + \dfrac{1/3}{s^2+1}\right]$.

so $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \dfrac{1}{3}\left[\sin(t) - \dfrac{\sin(2t)}{2}\right]$

$$+ \frac{1}{3} \times u_{2\pi}\left[\frac{\sin(2(t-2\pi))}{2} - \sin(t-2\pi)\right].$$

b)



5) a) $y''' + 2y' = 2x$.

$P_H(D) = D^3 + 2D = D(D^2+2)$.
$P_{NH}(D) = D^2$. ($\rightarrow$ Annihilator).
$\Rightarrow Q(D) = P_{NH}(D) \times P_H(D)$.

so $Q(D) = D^2(D^3 + 2D) = D^3(D^2+2)$.

b) $\rightarrow Q_1(D) = D^2 + 2$ such that $Q_1(D)y_1 = 0$.

so $y_1 = A\cos(\sqrt{2}\,x) + B\sin(\sqrt{2}\,x)$.

$P(D)y_1 = 2\sqrt{2}A\sin(\sqrt{2}x) - 2\sqrt{2}A\sin(\sqrt{2}x)$
$+ \left(-2\sqrt{2}B\cos(\sqrt{2}x)\right) + 2\sqrt{2}B\cos(\sqrt{2}x) = 0 \Rightarrow y_1$ is part of the homogeneous solution y_H

$\rightarrow Q_2(D) = D^3$ such that $Q_2(D)y_2 = 0$.

so $y_2 = Cx^2 + Dx + E$

$P(D)\,y_2 = 4Cx + 2D = 2x$ so $D = 0$, $C = 1/2$. $\Rightarrow E$ part of y_H

so $y = \underbrace{A\cos(\sqrt{2}x) + B\sin(\sqrt{2}x) + E}_{y_H} + \underbrace{\frac{x^2}{2}}_{y_p}$.

c) $y_H = \underbrace{A\cos(\sqrt{2}x)}_{y_1} + B\underbrace{\sin(\sqrt{2}x)}_{y_2} + E \times \underbrace{1}_{y_3}$

Variation of parameters:

$$W = \begin{vmatrix} \cos\sqrt{2}x & \sin\sqrt{2}x & 1 \\ -\sqrt{2}\sin\sqrt{2}x & \sqrt{2}\cos\sqrt{2}x & 0 \\ -2\cos\sqrt{2}x & -2\sin\sqrt{2}x & 0 \end{vmatrix} = 1 \times \begin{vmatrix} -\sqrt{2}\sin\sqrt{2}x & \sqrt{2}\cos\sqrt{2}x \\ -2\cos\sqrt{2}x & -2\sin\sqrt{2}x \end{vmatrix}$$

$$= 2\sqrt{2}\left(\sin^2(\sqrt{2}x) + \cos^2(\sqrt{2}x)\right)$$

$$W = 2\sqrt{2}$$

$y_p = C_1(x)y_1 + C_2(x)y_2 + C_3(x)y_3$.

$$C_1'(x) = \frac{\begin{vmatrix} 0 & \sin\sqrt{2}x & 1 \\ 0 & \sqrt{2}\cos\sqrt{2}x & 0 \\ 2x & -2\sin\sqrt{2}x & 0 \end{vmatrix}}{2\sqrt{2}} = -x\cos\sqrt{2}x.$$

so $C_1(x) = \left[-x\frac{\sin\sqrt{2}x}{\sqrt{2}}\right] + \int \frac{\sin\sqrt{2}x}{\sqrt{2}}dx = -x\frac{\sin\sqrt{2}x}{\sqrt{2}} - \frac{\cos\sqrt{2}x}{2}$.

$$C_2'(x) = \frac{\begin{vmatrix} \cos\sqrt{2}x & 0 & 1 \\ -\sqrt{2}\sin\sqrt{2}x & 0 & 0 \\ -2\cos\sqrt{2}x & 2x & 0 \end{vmatrix}}{2\sqrt{2}} = -x\sin\sqrt{2}x$$

so $C_2(x) = \left[x\frac{\cos\sqrt{2}x}{\sqrt{2}}\right] - \int \frac{\cos\sqrt{2}x}{\sqrt{2}}dx = x\frac{\cos\sqrt{2}x}{\sqrt{2}} - \frac{\sin\sqrt{2}x}{2}$

$$C_3'(x) = \frac{\begin{vmatrix} \cos\sqrt{2}x & \sin\sqrt{2}x & 0 \\ -\sqrt{2}\sin\sqrt{2}x & \sqrt{2}\cos\sqrt{2}x & 0 \\ -2\cos\sqrt{2}x & -2\sin\sqrt{2}x & 2x \end{vmatrix}}{2\sqrt{2}} = x. \quad \text{So } C_3(x) = \frac{x^2}{2}$$

so $y_p = -x\cos(\sqrt{2}x)\frac{\sin(\sqrt{2}x)}{\sqrt{2}} - \frac{\cos^2(\sqrt{2}x)}{2} + x\sin(\sqrt{2}x)\frac{\cos(\sqrt{2}x)}{\sqrt{2}} - \frac{\sin^2(\sqrt{2}x)}{2} + \frac{x^2}{2}$

$y_p = \frac{x^2}{2} - \frac{1}{2}$

d). $y_{p_b} = \frac{x^2}{2}$, $y_{p_c} = \frac{x^2}{2} - \frac{1}{2}$ which includes a part of y_H $\left(= -\frac{1}{2} \times y_3\right)$

- Variation of parameters was a bit more difficult because it involved 3×3 determinants.

6) a) The homogeneous equation is: $y'' - 3y' - 4y = 0$.

The characteristic polynomial is: $r^2 - 3r - 4 = 0$.

$r_1 = -1$ and $r_2 = +4$ are the 2 roots.

so $y_1(t) = e^{-t}$ and $y_2(t) = e^{4t}$.

and $y_H = Ae^{-t} + Be^{4t}$.

b) $Y(t) = v(t)\,y_1(t) = v(t)e^{-t}$.

so $Y' = v'(t)e^{-t} - v(t)e^{-t}$

$Y'' = v''(t)e^{-t} - v'(t)e^{-t} - v'(t)e^{-t} + v(t)e^{-t}$.

so in the equation:

$$v''(t)e^{-t} - 2v'(t)e^{-t} + \cancel{v(t)e^{-t}} - 3\left(v'(t)e^{-t} - \cancel{v(t)e^{-t}}\right) - \cancel{4v(t)e^{-t}} = 2e^{-t}.$$

so $v''(t)\cancel{e^{-t}} - 5v'(t)\cancel{e^{-t}} = 2\cancel{e^{-t}}$

$v''(t) - 5v'(t) = 2$

$\mu(t) = e^{\int -5dt} = e^{-5t}$ so $v'(t)e^{-5t} = \int 2e^{-5t}\,dt = -\frac{2}{5}e^{-5t}$.

so $v'(t) = -\frac{2}{5}$. so $v(t) = -\frac{2}{5}t$.

c) $y_p = Y(t) = -\frac{2t}{5}e^{-t}$.

d) $y = Ae^{-t} + Be^{4t} - \frac{2t}{5}e^{-t}$

$$\left.\begin{array}{l} y(0) = A + B = \alpha \\ y'(0) = -A + 4B - \frac{2}{5} = 0 \end{array}\right) \Longrightarrow \begin{array}{l} A + B = \alpha \\ A + 4B = 2/5 \end{array}$$

Using Cramer's Rule: $A = \dfrac{\begin{vmatrix} \alpha & 1 \\ 2/5 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -1 & 4 \end{vmatrix}} = \dfrac{4\alpha - 2/5}{4+1} = \dfrac{20\alpha - 2}{25}$

$$B = \dfrac{\begin{vmatrix} 1 & \alpha \\ -1 & 2/5 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -1 & 4 \end{vmatrix}} = \dfrac{2/5 + \alpha}{4+1} = \dfrac{2+5\alpha}{25}.$$

so $y = \dfrac{20\alpha - 2}{25}e^{-t} + \dfrac{2+5\alpha}{25}e^{4t} - \dfrac{2t}{5}e^{-t}$.

e) We want $y \xrightarrow[t\to\infty]{} 0$

we know that $e^{-t} \xrightarrow[t\to\infty]{} 0$ but $e^{4t} \xrightarrow[t\to\infty]{} \infty$.

so we want $\dfrac{2+5\alpha}{25} = 0$ so $\alpha = -\dfrac{2}{5}$.

7) Let $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ be the solution's form around x_0.

a) $x_0 = 0$ is an ordinary point

$$y' = \sum_{n=1}^{\infty} a_n \cdot n \cdot (x - x_0)^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)(x - x_0)^n$$

$$y'' = \sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1)(x - x_0)^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x - x_0)^n.$$

In $(1-x)y'' + y = 0$:

$$(1-x)\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

so $\sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=1}^{\infty} a_n x^n + 2a_2 + a_0 = 0$

so $a_2 = -\frac{a_0}{2}$; a_1

and $(n+2)(n+1)a_{n+2} - n(n+1)a_{n+1} + a_n = 0$

so $a_{n+2} = \dfrac{(n+1)n\,a_{n+1} - a_n}{(n+2)(n+1)}$

b) $\rightarrow$ Let $a_0 = 1$, $a_1 = 0$.

$$y_1 = 1 - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{24} + \cdots$$

$\rightarrow$ Let $a_0 = 0$, $a_1 = 1$.

$$y_2 = x - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{24} + \cdots$$

c) $$y_1' = -x - \frac{x^2}{2} - \frac{x^3}{8} + \cdots$$

$$y_2' = 1 - \frac{x^2}{2} - \frac{x^3}{3} - \frac{5x^4}{24} + \cdots$$

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

so y_1 and y_2 are independent and are therefore a fundamental set of solutions.

8) a) singular points at $t = 1$ and $t = -1$.

$\longrightarrow$ Equation in standard form: $u'' \underbrace{- \frac{2t}{1-t^2}}_{p(t)} u' + \underbrace{\frac{\alpha(\alpha+1)}{1-t^2}}_{q(t)} u = 0$.

$t = 1$:

$$\lim_{t\to 1}(t-1)p(t) = \lim_{t\to 1}\left(-\frac{2t(t-1)}{1-t^2}\right) = \lim_{t\to 1}\left(-\frac{2t(t-1)}{(1-t)(1+t)}\right) = \lim_{t\to 1}\left(\frac{2t}{1+t}\right) = 1 = p_0.$$

$$\lim_{t\to 1}(t-1)^2 q(t) = \lim_{t\to 1}\left(\frac{\alpha(\alpha+1)(t-1)^2}{(1-t^2)}\right) = \lim_{t\to 1}\left(\frac{\alpha(\alpha+1)(1-t)}{1+t}\right) = 0 = q_0.$$

so $t = 1$ is a regular singular point.

$\underline{t=-1}$:

$$\lim_{t\to -1} (t+1)\,p(t) = \lim_{t\to -1}\left(-\frac{2t(t+1)}{1-t^2}\right) = \lim_{t\to -1}\left(-\frac{2t\cancel{(t+1)}}{(1-t)\cancel{(t+1)}}\right) = 1 = p_0 .$$

$$\lim_{t\to -1} (t+1)^2 q(t) = \lim_{t\to -1}\left(\frac{\alpha(\alpha+1)(t+1)^2}{1-t}\right) = \lim_{t\to -1}\left(\frac{\alpha(\alpha+1)(t+1)}{(1-t)}\right) = 0 = q_0 .$$

so $t=-1$ is a regular singular point.

b) Letting $t-1=x$, or $x+1=t$, we have: $y(x) = u(x+1) = u(t)$.

$$\frac{d}{dx} f(x) = \frac{d}{dx} u(x+1) = \frac{d}{dx} u(t) = \frac{d}{dt} u(t) \times \overbrace{\frac{dt}{dx}}^{1} = \frac{d}{dt} u(t), \text{ similarly } \frac{d^2}{dx^2} f(x) = \frac{d^2}{dt^2} u(t).$$

so $-x(2+x)\,y''(x) - 2(x+1)\,y'(x) + \alpha(\alpha+1)\,y(x) = 0$

so $(x^2+2x)\,y''(x) + 2(x+1)\,y'(x) - \alpha(\alpha+1)\,y(x) = 0$.

this equation has 2 singular points: $x=0$, $x=-2$.

$\underline{x_0=0}$: $\displaystyle \lim_{x\to 0} x \times \frac{2(x+1)}{x(2+x)} = \lim_{x\to 0} \frac{2(x+1)}{2+x} = 1 = p_0 .$

$$\lim_{x\to 0} x^2 \frac{-\alpha(\alpha+1)}{x(2+x)} = \lim_{x\to 0} \frac{-x\alpha(\alpha+1)}{(2+x)} = 0 = q_0 .$$

So $x=0$ is a regular singular point.

c) at $x_0=0$: indicial equation $F(r) = r(r-1) + p_0 r + q_0 = r(r-1) + r = r^2$

so $r_{1,2}=0$. (repeated root).

d) $$a_n(r) = -\frac{[(n+r-1)p_1+q_1]a_{n-1} + \cdots + (rp_n+q_n)a_0}{F(n+r)}$$

$$a_n = -\frac{[(n-1)p_1+q_1]a_{n-1} + \cdots + q_n a_0}{F(n)} = -\frac{[(n-1)p_1+q_1]a_{n-1} + \cdots + q_n a_0}{n^2}$$

where $\displaystyle xp(x) = \frac{2(x+1)}{(2+x)} = \sum_{n=0}^{\infty} p_n x^n$ and $\displaystyle x^2 q(x) = -\frac{x\alpha(\alpha+1)}{(2+x)} = \sum_{n=0}^{\infty} q_n x^n$.

so $\displaystyle y_1(x) = \underbrace{|x|^r}_{1} \times \left[1 + \sum_{n=1}^{\infty} a_n x^n\right] = 1 + \sum_{n=1}^{\infty} a_n x^n$.

and $\displaystyle y_2(x) = y_1 \ln|x| + |x|^r \sum_{n=1}^{\infty} a_n'(r)\, x^n = y_1 \ln|x| + \sum_{n=1}^{\infty} a_n'(0)\, x^n$.